# $\sigma$-CONTINUITY AND RELATED FORCINGS

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Abstract. The Steprāns forcing notion arises as a quotient of Borel sets modulo the ideal of $\sigma$-continuity of a certain Borel not $\sigma$-continuous function. We give a characterization of this forcing in the language of trees and using this characterization we establish such properties of the forcing as fusion and continuous reading of names. Although the latter property is usually implied by the fact that the associated ideal is generated by closed sets, we show it is not the case with Steprāns forcing. We also establish a connection between Steprāns forcing and Miller forcing thus giving a new description of the latter. Eventually, we exhibit a variety of forcing notions which do not have continuous reading of names in any presentation.

## 1. Introduction

Many classical forcing notions arise as quotient Boolean algebras of $\mathrm{Bor}(X)$ modulo an ideal $I$ in a Polish space $X$. Forcings of this form are called idealized forcings (cf. [7]) and are usually denoted by $\mathbb{P}_I$ to indicate the ideal they arise from. In this way Cohen forcing is associated with the ideal of meager sets, Sacks forcing with countable sets and Miller forcing with $K_\sigma$ sets in the Baire space, to recall just a few examples. The generic extensions given by these forcings are always extensions by a single real, which is called the generic real.

Idealized forcings $\mathbb{P}_I$ often are equivalent to forcings with certain families of trees ordered by inclusion. For instance, Sacks forcing is the forcing with perfect trees and Miller forcing with superperfect trees in $\omega^{<\omega}$.

In examining forcing effects on the real line it is often convenient to have a nice representation of names for reals in the extension. In many examples such a representation is given by functions from the ground model. Namely each real in the extension is the value of a certain function from the ground model at the generic real. Not always, however, can the function be defined globally. If we assume properness of the forcing $\mathbb{P}_I$ then we are provided by a representation in terms of Borel functions:

**Theorem 1** (Zapletal, [7]). *If the forcing notion $\mathbb{P}_I$ is proper and $\dot{x}$ is a name for a real then for each $B \in \mathbb{P}_I$ there is a condition $C \leq B$ and a Borel function $f : C \to \mathbb{R}$ such that*

$$C \Vdash \dot{x} = f(\dot{g})$$

*where $\dot{g}$ is the name for generic real.*

The most desirable situation is when the function can be chosen to be continuous and in many cases it actually happens. This property is called continuous reading of names. One should be aware, however, that this property depends (at least formally) on the topology of the space $X$. How common this property is among idealized forcings, can be partially accounted for by the following theorem.

**Theorem 2** (Zapletal, [7]). *If the ideal $I$ is generated by closed sets then the associated forcing $\mathbb{P}_I$ is proper and has continuous reading of names.*

There is one important example of a forcing notion $\mathbb{P}_I$ which is proper but fails to have continuous reading of names in the natural topology of the space. Let us recall the old problem of Lusin whether there is a Borel function which is not $\sigma$-continuous. In [1] a particularly simple example of such a function was given, namely the Pawlikowski's function $P$. The ideal $I_P$ of sets on which $P$ is $\sigma$-continuous gives rise to the forcing notion $\mathbb{P}_{I_P}$, usually called (cf. [7]) the Steprāns forcing. In [5] Steprāns introduced this forcing and used it to increase the cardinal characteristic $\mathrm{cov}(I_P)$ in a generic extension. The key feature of the forcing $\mathbb{P}_{I_P}$ is that it adds a real which is not contained in any ground model set from $I_P$.

Since the ideal $I_P$ can be seen as a porosity ideal (cf. [7]), properness of the forcing follows from another general result.

**Theorem 3** (Zapletal, [7]). *If $I$ is a porosity ideal then the forcing $\mathbb{P}_I$ is proper.*

Steprāns forcing has many nice properties, one of them is the fact that compact sets are dense in it. This follows from the following theorem.

**Theorem 4** (Zapletal, [6]). *For any Borel not $\sigma$-continuous function $f : \omega^\omega \to \omega^\omega$ and for any Borel set $B \notin I_f$ there exists a compact set $C \subseteq B$ such that $C \notin I_f$.*

The proof of this theorem introduces a certain Borel game which detects $\sigma$-continuity of a given Borel function. The result follows then from determinacy of this game.

Steprāns forcing, however, does not have continuous reading of names, for the function $P$, treated as a name for a real, is itself a counterexample. This single obstacle may be handled by extending the topology to one which has the same Borel sets and makes $P$ continuous. A question is if this would bring about continuous reading of names in the Steprāns forcing.

This has been investigated by the authors of [2] who argued that the ideal associated with Steprāns forcing is generated by closed sets in the extended topology. This should result in continuous reading of names but the argument from [2] is incorrect. The problem whether the ideal is generated by closed sets in this extended topology was also raised in [5].

We will show that the ideal $I_P$ is not generated by closed sets in the extended topology. Nevertheless, we will prove that the Steprāns forcing has continuous reading of names in this topology. To this end we will establish a description of the forcing in terms of trees and deduce continuous reading of names from properties of these trees. This will also enable us to define fusion in Steprāns forcing.

In light of the above another question arises. Are there any forcing notions of the form $\mathbb{P}_I$ which do not have continuous reading of names in any presentation (i.e. in any Polish topology which gives the same Borel structure)? This question was already posed in [2]. Recently an example has been given by Zapletal in [7], namely he proved the eventually different real forcing has this property. We will present a different example.

In fact, we will show that such forcings are quite common among the idealized forcings. Namely, there is a method of constructing them out of forcings which, as the Steprāns forcing, do not have continuous reading of names in one topology.

## 2. Definitions and notation

Throughout this paper an ideal will always mean a $\sigma$-ideal of subsets of a Polish space.

In a space $X$ a system of sets indexed by a tree $T \subseteq Y^{<\omega}$ ($Y$ is an arbitrary set). is to be understood as a map $T \ni \tau \mapsto D_\tau \subseteq X$ such that if $\tau \subseteq \tau' \in T$ then $D_{\tau'} \subseteq D_\tau$. The system is disjoint if $D_\tau \cap D_{\tau'} = \emptyset$ for $\tau \neq \tau', |\tau| = |\tau'|$.

In a space $X^\omega$, whatever be its topology, for a finite partial function $\tau : \omega \to \mathcal{P}(X)$ we will denote by $[\tau]$ the set $\{t \in X^\omega : \forall n \in \mathrm{dom}(\tau)\ t(n) \in \tau(n)\}$. For a tree $T \subseteq X^{<\omega}$ let its limit, denoted $\lim T$, be the set $\{x \in X^\omega : \forall n \in \omega \quad x \restriction n \in T\}$. For a node

$\tau \in T$ the end-extension of $T$ above $\tau$ will stand for the subtree $\{\sigma \in T : \tau \subseteq \sigma \vee \sigma \subseteq \tau\}$. For a set $T_0 \subseteq T$ the end-extension of $T$ above $T_0$ will be the union of end-extensions of $T$ above elements of $T_0$.

We will say that a Borel function $f : X \to Y$, where $X, Y$ are Polish spaces, is $\sigma$-continuous if there exist a countable cover of the space $X = \bigcup_n X_n$ (with arbitrary sets $X_n$) such that $f \restriction X_n$ is continuous for each $n$. It follows from Kuratowski's extension theorem that we may require that the sets $X_n$ be Borel. If they can be chosen closed in $X$ then we shall say that $f$ is closed-$\sigma$-continuous.

If $f : X \to Y$ is not $\sigma$-continuous then $I_f$ denotes the ideal of sets on which $f$ is $\sigma$-continuous.

For two Borel functions $f : X \to Y$ and $f' : X' \to Y'$ we will say that $f$ can be factorized by $f'$ if there exist a continuous 1-1 function $\varphi : X' \to X$ and an open 1-1 $\psi : Y' \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} Y' & \xrightarrow{\psi} & Y \\ \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} \\ X' & \xrightarrow{\varphi} & X \end{array}$$

Obviously, if $f'$ is not $\sigma$-continuous and factorizes $f$ then $f$ is also not $\sigma$-continuous.

In a metric space $(X, d)$ for $A, B \subseteq X$ we will denote by $\mathrm{dist}(A, B)$ the infimum of $d(a, b)$ for $a \in A$ and $b \in B$. The Hausdorff distance between $A$ and $B$ will be denoted by $h(A, B)$.

The space $(\omega + 1)^\omega$ is endowed with the product topology of order topologies on $\omega + 1$. It is of course homeomorphic to the Cantor space. We also fix a metric $\rho$ on $(\omega + 1)^\omega$ which gives the above topology. For $x, y \in (\omega+1)^\omega$ let $\rho(x, y) = \sum_n \frac{1}{2^n} \rho'(x(n), y(n))$ where $\rho'$ metrizes $\omega+1$ with its order-topology, i.e. $\rho'(n, \omega) = \frac{1}{2^n}$ and $\rho'(n, m) = |\frac{1}{2^n} - \frac{1}{2^m}|$ for $n, m < \omega$. All metric notions on $(\omega + 1)^\omega$, like diameter, distance, etc., will be relative to the metric $\rho$.

The Pawlikowski's function $P : (\omega + 1)^\omega \to \omega^\omega$ is defined as follows:

$$P(x)(n) = \begin{cases} x(n) + 1 & \text{if} \quad x(n) < \omega, \\ 0 & \text{if} \quad x(n) = \omega. \end{cases}$$

It has been shown in [1] that $P$ is not $\sigma$-continuous. Hence $I_P = \{A \subseteq (\omega + 1)^\omega : P \restriction A \text{ is } \sigma\text{-continuous}\}$ is a proper ideal. It's subideal $I_P^c$ is defined analogously for closed-$\sigma$-continuity.

Note that the smallest topology on $(\omega+1)^\omega$ in which $P$ is continuous is the one with basic clopens of the form $[\sigma]$ for $\sigma \in (\omega + 1)^{<\omega}$. With

this topology $(\omega+1)^\omega$ is homeomorphic to the Baire space $\omega^\omega$. We will thus refer to the two topologies: the original and the extended one as Cantor and Baire topology, respectively.

Once we have an ideal in $\mathrm{Bor}(X)$ (the family of all Borel sets in $X$), we consider the associated forcing notion $\mathbb{P}_I$ which is the poset $(\mathrm{Bor}(X) \setminus I, \subseteq)$. Of course, it is equivalent to the Boolean algebra $\mathrm{Bor}(X)/I$. The Steprāns forcing, associated with $I_P$ will be denoted by $\mathbb{S}$ and the forcing associated with $I_P^c$ will be denoted by $\mathbb{S}_c$.

We will say that a forcing $\mathbb{P}_I$ has continuous reading of names in a topology $\mathcal{T}$ of the space $X$ if for any $B \in \mathbb{P}_I$ and any Borel function $f : B \to \omega^\omega$ there exists $\mathbb{P}_I \ni C \subseteq B$ such that $f \restriction C$ is continuous in $\mathcal{T}$.

The general definition of properness of a forcing notion can be found for instance in [3]. Let us, however, recall a characterization formulated by Zapletal in [7]: forcing of the form $\mathbb{P}_I$ is proper iff for every countable elementary substructure $M$ of a large enough $H_\kappa$ and every condition $B \in M \cap \mathbb{P}_I$ the set $\{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is not in $I$ (this set turns out to be always Borel).

Recall also that a forcing notion $\mathbb{P}$ satisfies Baumgartner's Axiom A if there is a sequence $\leq_n, n < \omega$ of partial orders on $\mathbb{P}$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\mathbb{P} \ni p_n, n < \omega$ are such that $p_{n+1} \leq_n p_n$ there is a $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all $n$,
- for every $p \in \mathbb{P}$, for every $n$ and for every ordinal name $\dot{x}$ there exist $\mathbb{P} \ni q \leq_n p$ and a countable set $B$ such that $q \Vdash \dot{x} \in B$.

Of course, forcings satisfying Axiom A are proper.

An ideal $I$ in a Polish space $X$ is said to be generated by closed sets if any Borel $B \in I$ has a $F_\sigma$ superset $C \in I$.

## 3. Characterization of the Steprāns forcing

Note first of all that for every closed set $C$ in the Cantor topology of $(\omega+1)^\omega$ there exists a tree $T \subseteq (\omega+1)^{<\omega}$ such that $C = \lim T$. Not for every tree, however, its limit is a closed set in the Cantor topology. In general, if $T \subseteq (\omega+1)^{<\omega}$ is a tree then $\lim T$ is a $G_\delta$ set in the Cantor topology (since its complement is the union of sets $[\tau]$ for $\tau \notin T$).

One very popular belief concerning the Steprāns forcing is that a Borel set $A$ is in $\mathbb{P}_I$ iff it contains limit of a tree $T \subseteq (\omega+1)^{<\omega}$ such that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many immediate successors including $\tau'^\frown\omega$. This is, however, not true, as we will show in the following example.

**Example.** We will now construct a tree $S$ with the property that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many

immediate successors including $\tau'^\frown\omega$ but $P$ is $\sigma$-continuous on $\lim S$. We will build the tree inductively on its levels. For any node $\tau \in S$ we will also define a set $A_\tau \subseteq \omega$. We begin with $\emptyset$ and put $A_\emptyset = \omega$. Suppose that we have the tree $S$ built up to level $k$. Now let each node $\tau$ split into $\tau^\frown\omega$ as well as $\tau^\frown n$ for $n \in A_\tau$. Define sets $A_{\tau^\frown i}$ for $i \in A_\tau \cup \{\omega\}$ so that they form a partition of $A_\tau$ into infinitely many infinite subsets. Note at this point that if $s \in \lim S$ and $s(n) < \omega$ then $s{\restriction}n$ is uniquely determined.

**Claim.** *The function $P$ is $\sigma$-continuous on* $\lim S$*.*

*Proof.* Define $X_n = \{s \in \lim S : \forall m \geq n \quad s(m) = \omega\}$ and $X_\infty = \lim S \setminus \bigcup_n X_n$. Note that $P$ is continuous on $X_\infty$. Indeed, take any convergent sequence $s_n \to s$ such that $s_n, s \in X_\infty$ and notice that if $s(m) < \omega$ then $s_n(m) = s(m)$ implies also $s_n \restriction m = s \restriction m$. Hence, since $[(m, s(m))]$ is a neighborhood of $s$, there exists $m' < \omega$ such that $s_n{\restriction}m = s{\restriction}m$ for $n > m'$. Thus, if $s$ has infinitely many values $< \omega$ then the sequence $s_n$ eventually stabilizes on each coordinate. This shows that also $P(s_n) \to P(s)$. Since all sets $X_n$ are countable, $P$ is $\sigma$-continuous on $\lim S$. □

Throughout the rest of this section all topological notions concerning the space $(\omega+1)^\omega$ will be relative to the Cantor topology.

As we have already mentioned, many forcings of the form $\mathbb{P}_I$ can be equivalently described as tree forcings. It turns out that Steprāns forcing has similar description in terms of subtrees of $(\omega+1)^{<\omega}$.

In fact, the forcing $\mathbb{SP}$ considered by Steprāns in [5] is actually a forcing with trees. It adds an $\mathbb{S}$-generic but it is not clear that $\mathbb{SP}$ is equivalent to $\mathbb{S}$. But since $\mathbb{SP}$ can be easily seen as a dense subset of $\mathbb{Q}$ (see below), the equivalence follows from Theorem 5.

**Definition 1.** A tree $T \subseteq (\omega+1)^{<\omega}$ will be called wide if every node $\tau \in T$ has an extension $\tau' \in T$ such that the set $\lim T \cap [\tau']$ is nowhere dense in $\lim T \cap [\tau]$. A subset of $(\omega+1)^\omega$ will be called wide if it is the limit of a wide tree.

Obviously, the node $\tau'$ above must be of the form $\tau''^\frown\omega$ for some $\tau'' \supseteq \tau$ which splits in $T$ into infinitely many immediate successors.

Let us denote by $\mathbb{Q}$ the poset of wide trees ordered by inclusion.

**Theorem 5.** *The Steprāns forcing is equivalent to the forcing* $\mathbb{Q}$*.*

The idea to consider wide sets comes from the proof of the famous theorem of Solecki.

**Theorem 6** (Solecki, [4])**.** *For any Baire class 1 function $f : X \to Y$, where $X, Y$ are Polish spaces, either $f$ is $\sigma$-continuous or else there*

*exist topological embeddings $\varphi$ and $\psi$ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & Y \\ \uparrow P & & \uparrow f \\ (\omega+1)^\omega & \xrightarrow{\varphi} & X \end{array}$$

*Proof of theorem 5.* The assertion results from Proposition 1 and Proposition 2 given below. □

**Proposition 1.** *Assume $B \subseteq (\omega+1)^\omega$ is Borel such that $P{\restriction}B$ is not $\sigma$-continuous. Then there exists a wide tree $T$ such that $\lim T = D \subseteq B$.*

*Proof.* Let us begin with a claim.

**Claim.** *Let $E \subseteq (\omega+1)^\omega$ be closed such that $P{\restriction}E$ is not continuous. There exists a sequence of disjoint relative clopens $C_n, n < \omega$ each of the form $[\tau_n]$ such that $\bigcup_n C_n$ is dense in $E$.*

*Proof.* Note that any relative open set in $E$ contains a relative clopen of the form $[\sigma]$, where $\sigma \in (\omega+1)^{<\omega}$. This follows from the fact that any basic open set (in $E$) of the form $[\tau_1^\frown[n,\omega]^\frown\tau_2]$ either has a nonempty (in $E$) clopen subset of the form $[\tau_1^\frown m^\frown\tau_2]$ for some $m \geq n$ or is equal to the relative clopen $[\tau_1^\frown\omega^\frown\tau_2]$. Take thus a maximal antichain of clopens (in $E$) of the form $[\sigma]$. This antichain can be taken infinite for $E$ is limit of a tree which is not finitely-branching (otherwise $P$ would be continuous on $E$) and thus we may extend an antichain given by infinitely many immediate successors (by numbers less than $\omega$) of a chosen node. □

**Definition 2.** A fusion system in $(\omega+1)^\omega$ is a tree $T \subseteq (\omega+1)^{<\omega}$ together with a family of trees $T_\tau, \tau \in T$ such that

- each $\lim T_\tau$ is closed,
- $T_\tau$ has stem $\tau$,
- for $\tau \subseteq \tau' \in T$ $\ T_{\tau'} \subseteq T_\tau$.

Now we pass to the main proof. By Theorem 4 we may assume that $B = \lim T_\emptyset$ is closed. We may also assume that $P$ is not $\sigma$-continuous on any basic clopen set. We will construct a fusion system $T \subseteq (\omega+1)^{<\omega}$, $T_\tau, \tau \in T$ (and denote $D_\tau = \lim T_\tau$) so that the set $D = \lim T$ will be wide.

The construction is carried out inductively (beginning with $T_\emptyset$) in such a way that having constructed $\tau$ and $T_\tau$ we find infinitely many (pairwise incomparable) extensions of $\tau$ and appropriate family of subtrees of $T_\tau$.

Suppose we have constructed a node $\tau$ and a tree $T_\tau$. By the above Claim we can find an antichain $\tau_n, n<\omega$ of extensions of $\tau$ such that $\{D_\tau \cap [\tau_n] : n<\omega\}$ is a maximal antichain of relative clopens in $D_\tau$. We put $T_{\tau_n}$ to be the end-extension above $\tau_n$ in $T$. Let us look now at the closed set $E = D_\tau \setminus \bigcup_n D_{\tau_n}$. In case $P$ is $\sigma$-continuous on this set we will extend $\tau$ by $\tau_n$'s only and call this extension regular. If, however, $P$ is not $\sigma$-continuous on $E$ then let us first shrink it to $E'$ by cutting off all relative clopens on which $P$ is $\sigma$-continuous. Then take any $\tau_\omega \in (\omega+1)^{<\omega}$ which gives a nonempty relative clopen in $E'$ (of length $> |\tau|$). Now extend $\tau$ additionally by $\tau_\omega$ as well as define $T_{\tau_\omega}$ as the tree of $E' \cap [\tau_\omega]$. The extension of this form will be called irregular and we will refer to $\tau_\omega$ as the irregular node.

Once the tree has been constructed let us note that each node $\tau$ has an irregular extension $\tau'$. Indeed, for otherwise $D_\tau$ would be a union of countably many closed sets which we have cut off (on each of which $P$ was $\sigma$-continuous) and the set $[\tau] \cap \lim T$. On the latter set, however, $P$ is continuous, hence we would get that $P$ is $\sigma$-continuous on $D_\tau$, which is not the case. So what is left is to show that the set $D_{\tau'} \cap D$ is nowhere dense in $D_\tau \cap D$. But its complement contains the intersection of a sequence of unions of relative clopens which occur either in regular extensions or in irregular as those non-irregular ones. This is, however, an intersection of a sequence of dense open sets and it is dense by the Baire category theorem (recall that $D$ is a $G_\delta$). So the image of the irregular node is nowhere dense, as claimed. $\square$

**Remark 1.** The fusion method from the above proof will be further used to established Axiom A and continuous reading of names. We would like to mention, however, that Proposition 1 can be also proved without fusion, using the method of Cantor-Bendixson analysis instead: Call a tree $T \subseteq (\omega+1)^{<\omega}$ small if for each $\tau \in T$ the set $[\tau^\frown\omega]$ is relatively open in $\lim T$. It is easy to see that if $T$ is small then $P$ is continuous on $\lim T$. Then use a procedure in the fashion of the Cantor-Bendixson analysis to cut off from $T$ all nodes (and their extensions) such that the end-extension above them is small. Then the remaining tree will be obviously wide. It is maybe more clear now that the wide set which remains is not in the ideal $I_P$. But we do not need this because Proposition 2 says that actually any wide set is $I_P$-positive.

**Remark 2.** Yet another way of proving Proposition 1 is to apply Theorem 6 together with Theorem 4 and the fact that $P$ is of Baire class 1. But the arguments given above are much simpler than the proof of Theorem 6 in its full strength. Nevertheless, one of the ideas from Solecki's proof of Theorem 6 will be used in the proof of Proposition 2.

**Proposition 2.** *Assume $D \subseteq (\omega+1)^\omega$ is a wide set. Then there are topological embeddings $\varphi$ and $\psi$ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & P[D] \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\varphi} & D \end{array}$$

*Proof.* Let us suppose $D = \lim T$ is wide in $(\omega+1)^\omega$. Let us say that a subtree $S \subseteq T$ is an end-subtree if there is a finite set of nodes of $T$ such that $S$ is the end-extension of this set. Let $\mathbb{C}$ be the forcing with end-subtrees of $T$ ordered by inclusion. $\mathbb{C}$ is of course equivalent to the Cohen forcing. Let $M$ be a countable elementary submodel of a large enough $H_\kappa$ such that $P, \mathbb{C}, D \in M$. Let $\mathcal{D}_n, n < \omega$ enumerate all dense subsets of $\mathbb{C}$ in $M$. For a dense set $\mathcal{D} \subseteq \mathbb{C}$ let $\mathcal{D}^*$ denote the set of all finite unions of elements from $\mathcal{D}$.

We are going to construct only the embedding $\varphi : (\omega+1)^\omega \to D$ since it already determines the function $\psi$. To this end we will define a disjoint system of wide $G_\delta$ sets $D_\tau \subseteq D, \tau \in (\omega+1)^{<\omega}$ given as limits of trees $T_\tau \in \mathbb{C}$, such that the branches of the system (i.e. $\{T_\tau : \tau \subseteq t\}$ for $t \in (\omega+1)^\omega$) will generate $\mathbb{C}$-generic filters over $M$.

The trees $T_\tau$ will be constructed by induction on $|\tau|$ and will satisfy the following conditions:

(i) $T_\tau \in \mathcal{D}^*_{|\tau|}$,
(ii) $\operatorname{diam}(D_\tau) < 1/|\tau|$,
(iii) for each $n$ the map $(\omega+1)^n \ni \tau \mapsto D_\tau$ is $h$-continuous.

Notice that (i) and (ii) implies that each branch generates a generic filter over $M$. Indeed, because any extension to an ultrafilter must be generic by (i) and by (ii) there is precisely one such extension, since it is determined by an appropriate generic real.

Note at this point that the sets $D_\tau$ suffice to construct a factorization. For $t \in (\omega+1)^\omega$ we define $\varphi(t)$ to be the generic real given by the generic filter along the branch $t$. Thanks to (iii) $\varphi$ is continuous. Because of disjointness of the system, $\varphi$ is injective, and hence a topological embedding. On the other hand, $\psi$ is open because the system is disjoint and $P[D_\tau]$ is open in $P[D]$ (since $T_\tau$ is an end-subtree). To see that $\psi$ is continuous we use genericity: a formula of the form $P(\varphi(t))(m) = n$ is absolute for transitive models and if it holds for $t$ which is generic over $M$ then it must be forced by some condition in the generic filter.

Before we go on and construct the sets $D_\tau$, let us first endow the space $(\omega+1)^n$ with some additional structure which will be used in

the construction. Let $S^n_k \subseteq (\omega+1)^n$ be the set of points of Cantor-Bendixson rank $\geq n-k$ (for $k \leq n$). Besides the sets $S^n_k$, let us also define a system of projections $\pi^n_k : S^n_k \to S^n_{k-1}$ for $1 \leq k \leq n$. If we embed $(\omega+1)^n$ into the cube $[0,1]^n$ via $0 \mapsto 0$, $n \mapsto 1-1/n$ for $n > 0$ then $S^n_k$ can be viewed a set of $\binom{n}{k}$ its $k$-dimensional faces of the cube. In this setting each $S^n_k$ can be projected orthogonally onto $S^n_{k-1}$. This projection, however, is ambigous at some diagonal points. Nevertheless, we may pick one of the possible values and in this way define a function $\pi^n_k$. In other words, if $\tau \in S^n_k \setminus S^n_{k-1}$ then we pick one $i \in n$ such that $\tau(i)$ is maximal value less than $\omega$ and define $\pi^n_k(\tau)(i) = \omega$ and $\pi^n_k(\tau)(j) = \tau(j)$ for $j \neq i$. On $S^n_{k-1}$ $\pi^n_k$ is the identity. The key feature of these functions is that they are continuous, no matter which values we have picked.

**Lemma 1.** *For each $n$ and $1 \leq k \leq n$ the projection $\pi^n_k : S^n_k \to S^n_{k-1}$ is continuous.*

*Proof.* Note that any point in $S^n_k$ except $(\omega,\dots,\omega)$ ($k$ times $\omega$) has a neighborhood in which projection is unambigous and hence continuous. But it is easy to see that at the point $(\omega,\dots,\omega)$ any projection is continuous. □

In the construction we will use the following lemma which holds in $M$.

**Lemma 2.** *Let $S, S'$ be end-subtrees of $T$, $D = \lim S, D' = \lim S'$, $\delta > 0$ and $k < \omega$.*

1. *There is a sequence $S_i, i \in \omega+1$ of subtrees of $S$ such that $S_i \in \mathbb{C}$, the sets $D_i = \lim S_i$ are disjoint, the map $\omega{+}1 \ni i \mapsto D_i$ is h-continuous and for each $i$ diam$(D_i) < \delta$ and $D_i \in \mathcal{D}^*_k$.*
2. *If $S_i$ are as above then there is a sequence $S'_i, i \in \omega{+}1$ of subtrees of $S'$ such that $S'_i \in \mathbb{C}$, the sets $D'_i = \lim S'_i$ are disjoint, the map $\omega+1 \ni i \mapsto D'_i$ is h-continuous, for each $i$ $D'_i \in \mathcal{D}^*_k$, diam$(D'_i) < 3\delta$ and $h(D_i, D'_i) \leq 3h(D, D')$.*

*Proof.* (1) Let us pick any node $\tau^\omega \in S$ such that $|\tau^\omega| > k$, $\lim S \cap [\tau^\omega]$ has diameter $< \delta/3$ and is nowhere dense in $\lim S$ (let $\tau^\omega = \tau^\frown\omega$). Put $S_\omega$ equal to the end-subtree of $S$ above $\tau^\omega$. Notice that for any $\varepsilon > 0$ there exists a finite set $\tau_i, i \leq n$ of extensions of $\tau^\omega$ such that $\mathrm{diam}(D_\omega \cap [\tau_i]) < \varepsilon$ for each $i \leq n$ and $h(D_\omega, \bigcup_{i\leq n} D_\omega \cap [\tau_i]) < \varepsilon$. Using the fact that $\lim S \cap [\tau^\omega]$ is nowhere dense in $\lim S$ we can find nodes $\tau'_i$ (being extensions of nodes $\tau^\frown n_i$ for some $n_i < \omega$) such that $\mathrm{diam}(D \cap [\tau'_i]) < \varepsilon$ and $\mathrm{dist}(D \cap [\tau'_i], D_\omega \cap [\tau_i]) < \varepsilon$ hold for each $i \leq n$. Now it easily follows that $h(D_\omega, \bigcup_{i\leq n} D \cap [\tau'_i]) < 3\varepsilon$ (so in particular

the first set has diameter $< \delta$ if $\varepsilon$ is small enough). We may of course shrink each $D \cap [\tau_i']$ so that it is the limit of a tree in $\mathcal{D}_k$. This ends the first part of the proof.

(2) Let $\gamma = h(D, D')$. First we claim that there is a finite set of nodes $\tau_i', i \leq n$ in $S'$ such that

- $\mathrm{diam}(D' \cap [\tau_i']) < \gamma$ for each $i \leq n$,
- $h(D_\omega, \bigcup_{i\leq n} D' \cap [\tau_i']) < 2\gamma$,
- $\mathrm{diam}(\bigcup_{i\leq n} D' \cap [\tau_i']) < 3\delta$.

Indeed, if $\gamma < \delta$ then we may first find finitely many nodes $\tau_i, i \leq n$ in $S_\omega$ and then appropriate nodes $\tau_i', i \leq n$ in $S'$ such that $h(D_\omega, \bigcup_{i\leq n} D_\omega \cap [\tau_i]) < 2\gamma$, both $\mathrm{diam}(D_\omega \cap [\tau_i]), \mathrm{diam}(D' \cap [\tau_i']) < \delta$ and also $\mathrm{dist}(D_\omega \cap [\tau_i], D' \cap [\tau_i']) < \delta$ for $i \leq n$. Then by the triangle inequality $\mathrm{diam}(\bigcup_{i\leq n} D' \cap [\tau_i']) < 3\delta$. If $\gamma \geq \delta$ then we will do by picking in a similar manner just one node $\tau$ and $\tau'$ in $S, S'$ respectively.

Now above each $\tau_i'$ choose a node $\tau_i^{\omega\prime}$ such that $D' \cap [\tau_i^{\omega\prime}]$ is nowhere dense in $D'$. Then $\bigcup_{i\leq n} D' \cap [\tau_i^{\omega\prime}]$ is also nowhere dense and has diameter $< 3\delta$. We may now find conditions in $\mathcal{D}_k$ which are stronger than the end-extensions in $S'$ above the nodes $\tau_i^{\omega\prime}$. And put $S_\omega'$ to be the union of these, $D_\omega' = \lim S_\omega'$. It is clear that $h(D_\omega, D_\omega') < 3\gamma$. Now as in (1) we can find a sequence of disjoint subtrees $S_n' \in \mathcal{D}_k^*$ such that if we put $D_n' = \lim S_n'$ then $h(D_n', D_\omega) < 1/n$. Now it follows from the triangle inequality that $h(D_n, D_n') < 3\gamma$ as well as $\mathrm{diam}(D_n') < 3\delta$ holds for $n$ big enough. But we may change those finitely many $S_n$'s (in the same way we have found $S_\omega'$, possibly shrinking the existing sets) to ensure that this holds for all $n$. $\square$

Now we proceed as follows. Let $D_\emptyset = D$. Having defined $D_\tau$ for all $\tau \in (\omega+1)^n$ we define it for $\tau \in (\omega+1)^{n+1}$. This in turn is done by another induction on the sets $S_k \times (\omega+1)$ for $0 \leq k \leq n$. That is, we first define $D_\tau$ for $\tau \in S_0^n \times (\omega+1)$ and then show how to extend the definition from $S_k^n \times (\omega+1)$ to $S_{k+1}^n \times (\omega+1)$. During this construction we take care that for each $k$ and $\tau \in S_k^n$

$$\mathrm{diam}(D_\tau) < 1/(3^{n-k}(n+1)) \tag{1}$$

and the map $S_k \times (\omega+1) \ni \tau \mapsto D_\tau$ is $h$-continuous.

To start with we use Lemma 2(1). Suppose we have $D_\tau$ defined for $\tau \in S_k \times (\omega+1)$. Let us abbreviate $\pi_k^n : S_{k+1}^n \to S_k^n$ by $\pi$ for a moment. For each $\tau \in S_{k+1}^n$ we use Lemma 2(2) for $D_\tau$ and $D_{\pi(\tau)}$ to find sets $D_{\tau^\frown i}$ for $i \in \omega+1$ such that

$$h(D_{\tau^\frown i}, D_{\pi(\tau)^\frown i}) \leq 3h(D_\tau, D_{\pi(\tau)}), \tag{2}$$

$\operatorname{diam}(D_{\tau^\frown i}) < 3\operatorname{diam}(D_\tau)$ and $D_{\tau^\frown i} \in \mathcal{D}^*_{|\tau|}$. Now (1) follows from the inductive assumption. To see $h$-continuity notice that if $(\tau_n, i_n) \to (\tau, i)$ is a convergent sequence in $S^n_{k+1} \times (\omega+1)$ then either $\tau_n$ is eventually constant, in which case the continuity is easy, or $\tau \in S^n_k$ and then $\pi(\tau_n) \to \tau$ thanks to the continuity of $\pi$. But then the assertion follows from the induction assumption, (2) and the triangle inequality:

$$h(D_{\tau_n^\frown i_n}, D_{\tau^\frown i}) \leq h(D_{\tau_n^\frown i_n}, D_{\pi(\tau_n)^\frown i_n}) + h(D_{\pi(\tau_n)^\frown i_n}, D_{\tau^\frown i_n}) \\ + h(D_{\tau^\frown i_n}, D_{\tau^\frown i}).$$

In this way we have constructed the sets $D_\tau$ and finished the proof. □

Propositions 5 and 2 have the following corollaries.

**Corollary 1.** *If $B \subseteq (\omega+1)^\omega$ is a Borel set such that $P \restriction B$ is not $\sigma$-continuous then there exists a closed wide set $D \subseteq B$.*

*Proof.* This follows from Proposition 2 since the image of $\varphi$ is a wide closed set. □

The second corollary is a particular case of Theorem 6 when $f$ is the restriction of $P$ to a $I_P$-large set.

**Corollary 2.** *If $D \subseteq (\omega+1)^\omega$ is Borel then either $P\restriction D$ is $\sigma$-continuous or else there are topological embeddings $\varphi$ and $\psi$ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & P[D] \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\varphi} & D \end{array}$$

## 4. Continuous reading of names

Let us recall now that Steprāns forcing does not have continuous reading of names in the Cantor topology of $(\omega+1)^\omega$. We will now show that it has continous reading of names in the Baire topology.

**Theorem 7.** *The forcing notion $\mathbb{S}$ has continuous reading of names in the Baire topology on $(\omega+1)^\omega$.*

Recall that all metric notions on $(\omega+1)^\omega$ (like diameter, distance, etc.) are relative to the metric $\rho$ (see Section 2) on $(\omega+1)^\omega$.

*Proof.* Let $B$ be any Borel $I_P$-positive set in $(\omega+1)^\omega$ and $\dot{x}$ be a $\mathbb{S}$-name for a real. By Proposition 1 we may assume $B$ is a limit of a wide tree. Continuous reading of names will result from the following claim.

**Claim.** *Let $T$ be a tree and $\sigma \in T$ be such that $[\sigma^\frown\omega] \cap \lim T$ is nowhere dense in $[\sigma] \cap \lim T$. Then for each $\tau \in T$ such that $\sigma^\frown\omega \subseteq \tau$, any $\varepsilon > 0$ and $n < \omega$ there is $m > n$ and $\sigma^\frown m \subseteq \tau' \in T$ such that*

- *$[\tau'] \cap \lim T$ is a relative clopen,*
- *$diam([\tau'] \cap \lim T) < \varepsilon/2$,*
- *$dist([\tau] \cap \lim T, [\tau'] \cap \lim T) < \varepsilon/2$.*

*Proof.* Consider the family of relative clopen sets of the form $[\tau'] \cap \lim T$ with $\tau'$ extending some $\sigma^\frown m, m > n$ and having diameters $< \varepsilon/2$. Put also $\delta = \inf_{i \leq n} \operatorname{dist}([\tau], [\sigma^\frown i])$. If the assertion of this lemma were false, then the open ball around $[\tau]$ with radius $\min\{\delta, \varepsilon/2\}$ would exhibit that $\sigma^\frown\omega$ has nonempty interior. □

Now let us finish the proof of the theorem. We will say that a set of nodes of a tree $S$ is a spanning set if $S$ is the smallest tree containing those nodes. We will find a wide subtree $S \subseteq T$ and a spanning set $\{\tau_\sigma : \sigma \in \omega^\omega\}$ (with $\sigma \mapsto \tau_\sigma$ being order isomorphism) of nodes of $S$ together with a set of natural numbers $k_\tau$ such that for $\tau$ in the spanning set

$$(\lim S \cap [\tau]) \Vdash_{\mathbb{S}} \dot{x}(|\tau|) = k_\tau.$$

This will clearly show that on $\lim S$ the name $\dot{x}$ is read by a function continuous in the Baire topology relativized to $\lim S$.

The construction is conducted inductively on $|\tau|$ in a fusion manner, that is we define additionally subtrees $T_\tau$ with stems $\tau$, respectively. Suppose we have found everything up to the level $n$. We will show how to extend a single node. First find an extension $\tau \subseteq \tau' \in T_\tau$ such that $[\tau'^\frown\omega] \cap T_\tau$ is nowhere dense in $[\tau] \cap T_\tau$. Without loss of generality let us assume that $\{n < \omega : \tau'^\frown n \in T_\tau\}$ is the whole $\omega$. Now find a forcing extension of $[\tau'^\frown\omega] \cap T_\tau$ to a limit of a wide tree $T_{\tau^\frown 0}$ such that for some $k_{\tau^\frown 0}$

$$\lim T_{\tau^\frown 0} \Vdash_{\mathbb{S}} \dot{x}(|\tau| + 1) = k_{\tau^\frown 0}.$$

Next using the above lemma and some bookkeeping find extensions $\tau'^\frown n \subseteq \tau'_n, n \in \omega$ so that for any $\sigma \in T_{\tau^\frown 0}$ and $n < \omega$ there is $m \in \omega$ such that $\operatorname{dist}(\lim T_\tau \cap [\tau'_m], \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/n$ and $\operatorname{diam}(\lim T_\tau \cap [\tau'_m]) < 1/n$. Then extend the forcing conditions $[\tau'_m] \cap \lim T_\tau$ to limits of wide trees $T_{\tau^\frown m+1}$ such that for some natural numbers $k_{\tau^\frown m+1}$

$$\lim T_{\tau^\frown m+1} \Vdash_{\mathbb{S}} \dot{x}(|\tau| + 1) = k_{\tau^\frown m+1}.$$

Notice that if

$$\operatorname{diam}(\lim T_\tau \cap [\tau'_m]),\ \operatorname{dist}(\lim T_\tau \cap [\tau'_m], \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/2n$$

then

$$\operatorname{dist}(\lim T_{\tau^\frown m+1}, \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/n,$$

so the interior of $T_{\tau^\frown 0}$ remains empty. Moreover, it will remain empty even when we pass to the fusion tree $S$. Thus after the fusion we get an $I_P$-positive set and numbers $k_\tau$ which define a continuous function in the Baire topology. $\square$

## 5. The fusion

It has been established by Zapletal both in [7] and in [6] that Steprāns forcing is proper. The fusion method used in Theorem 5 and Theorem 7 suggests, however, that Axiom A can be deduced quite easily once we have the notion of a wide tree. Axiom A was also established by Steprāns for the forcing $\mathbb{SP}$ considered in [5]. We present a different proof that seems more natural for the forcing of wide trees.

**Theorem 8.** *The Steprāns forcing notion satisfies Axiom A.*

*Proof.* Let $\mathbb{W}'$ be a forcing with trees $T$ satisfying the following conditions:

(1) each $\tau \in T$ either has only one immediate successor or is such that $\tau^\frown\omega \in T$ and $[\tau^\frown \lim T] \cap \lim T$ is nowhere dense in $\lim T$,
(2) whenever $\tau \in T$ is such that $\tau^\frown\omega \in T$ and $[\tau^\frown \lim T] \cap \lim T$ is nowhere dense in $\lim T$, we have the following. For each $n < \omega$ such that $\tau^\frown n \in T$ denote the stem of the tree $T$ above $\tau^\frown n$ by $\tau_n$. For each clopen $C$ intersecting $\lim T \cap [\tau^\frown\omega]$ and any $\varepsilon > 0$ there is $n < \omega$ such that $\mathrm{diam}[\tau_n] < \varepsilon$ and $\mathrm{dist}([\tau_n], C) < \varepsilon$.

It is easy to see that $\mathbb{W}'$ is dense in $\mathbb{W}$. So it is enough to show that $\mathbb{W}'$ satisfies Axiom A.

Let $\prec$ be a linear order on $(\omega+1)^{<\omega}$ such that each $\tau$ occurs later than its initial segments. For a tree $T \in \mathbb{W}'$ let $w(T)$ be the set of those nodes of $T$ which have more than one immediate successor. For $\tau \in w(T)$ the set of its *immediate successors in* $w(T)$ stands for the set $\{\tau' \in w(T) : \neg\exists \tau'' \in w(T) \quad \tau \subsetneq \tau'' \subsetneq \tau'\}$. Now let us denote by $w_n(T)$ the set of $n$ first (with respect to $\prec$) elements of $w(T)$ together with theirs immediate successors in $w(T)$.

For $T, S \in \mathbb{W}'$ let $T \leq_n S$ if $T \leq S$ and $w_n(S) \subseteq T$. It is now easy to see that with these orderings $\mathbb{W}'$ satisfies Axiom A. $\square$

## 6. Generating by closed sets

Both properness and continuous reading of names could be deduced more easily if only we knew that $I_P$ were generated by closed sets in the Baire topology. A typical mistake that may lead to such a conclusion is the conviction that if $A \subseteq (\omega+1)^\omega$ is such that $P{\restriction}A$ is continuous in the Baire topology then so it is on the closure of $A$ in the Baire

topology. This is not true, as observed by Pawlikowski (in a private conversation).

The next proposition says that continuous reading of names can hold even when the ideal is not generated by closed sets.

**Proposition 3.** *The ideal $I_P$ is not generated by closed sets in the Baire topology.*

Throughout this proof let $\overline{X}$ (for $X \subseteq (\omega+1)^\omega$) denote the closure of $X$ in the Baire topology.

*Proof.* Let us first consider the following set $A = \{\alpha_n, \beta_n : n < \omega\} \subseteq (\omega+1)^\omega$, where

$$\alpha_n(0) = n, \quad \alpha_n(k) = \omega \text{ for } k > 0$$

$$\beta_n(n) = 0, \quad \beta_n(k) = \omega \text{ for } k \neq n.$$

Note that $P \restriction A$ is continuous. On the other hand, $P \restriction \overline{A}$ is not continuous since $\alpha = (\omega, \omega, \ldots) \in \overline{A}$ and $\alpha_n \to \alpha$, whereas $P(\alpha_n) \not\to P(\alpha)$.

Using a bijection from $\omega$ to $\omega \times \omega$ we may identify $(\omega+1)^\omega$ with $(\omega+1)^{\omega\times\omega} \simeq ((\omega+1)^\omega)^\omega$. Under this identification $P$ becomes $\prod_{n<\omega} P$, which we will denote by $P^\omega$. First note that $P^\omega$ is continuous on $A^\omega$ as a product of continuous functions, so $A^\omega \in I_{P^\omega}$. We will prove, however, that $A^\omega$ cannot be covered by countably many sets $F_n, n < \omega$ closed in the Baire topology, with each $F_n \in I_{P^\omega}$.

Suppose that $A^\omega \subseteq \bigcup_n F_n$ and $F_n$ are closed in the Baire topology. As $A$ is a discrete set in $(\omega+1)^\omega$ (in both topologies), the relative topology (with respect to any of these two) on $A^\omega$ is that of the Baire space. $F_n \cap A^\omega$ are relatively closed, so by the Baire category theorem, one of them has nonempty interior. This means that there is $n < \omega$, $k < \omega$ and $\alpha \in A^k$ such that $\alpha^\frown A^{\omega\setminus k} \subseteq F_n$. Without loss of generality $k = 0$ and $\overline{A^\omega} \subseteq F_n$. But $\overline{A^\omega} = (\overline{A})^\omega$ and $\overline{A}$ contains a convergent sequence $\alpha_n \to \alpha$ such that $P(\alpha_n) \not\to P(\alpha)$. So if $A' = \{\alpha, \alpha_n : n < \omega\}$ then $P[A']$ is a discrete set and $(A')^\omega \subseteq F_n$. Notice, however, that $P^\omega \restriction (A')^\omega = (P \restriction A')^\omega$ is not $\sigma$-continuous, since it can obviously be factorized by $P$. Hence $F_n \notin I_{P^\omega}$, which ends the proof. $\square$

## 7. Connections with the Miller forcing

A natural question that arises after realizing that Steprāns forcing is described in terms of wide trees is whether this forcing is equivalent to the Miller forcing. A negative answer follows for instance from Proposition 5 below. It turns out, however, that Miller forcing is very close to $\sigma$-continuity, namely it is isomorphic to the forcing associated to the ideal of closed-$\sigma$-continuity of $P$.

**Proposition 4.** *The forcing notion $\mathbb{S}_c$ is equivalent to the Miller forcing notion.*

*Proof.* The Miller forcing is equivalent to the forcing $\mathrm{Bor}(\omega^\omega)/K_\sigma$ and $\mathbb{S}_c$ is equivalent to $\mathrm{Bor}((\omega+1)^\omega)/I_P^c$. The isomorphism is given by the function $P$ itself (as it gives rise to a Borel isomorphism of the spaces). The only thing to realize is that for $A \subseteq (\omega+1)^\omega$ $P[A]$ is compact if and only if $A$ is a closed set on which $P$ is continuous. But this is the case since a continuous image of a compact set is compact, $P^{-1}$ is continuous and a continuous bijection defined on a compact set is a homeomorphism. $\square$

The following definition is a natural generalization of well known notions like Cohen real, Miller real, etc.

**Definition 3.** Let $M$ be a transitive countable model. We say that $s \in (\omega+1)^\omega$ is a Steprāns real over $M$ if $s \notin A$ for any $A \subseteq (\omega+1)^\omega$ such that $A \in I_P$ and $A$ is coded in $M$.

It is obvious that the generic real for Steprāns forcing is a Steprāns real over the ground model. In order to distinguish Steprāns forcing from Miller forcing let us prove that there are no Steprāns reals in extensions by a single Miller real.

**Proposition 5.** *Miller forcing does not add Steprāns real.*

*Proof.* Let us denote the Miller forcing notion by $\mathbb{M}$. Suppose, towards a contradiction, that $\dot{s}$ is a $\mathbb{M}$-name for a Steprāns real. Since $(\omega+1)^\omega \simeq 2^\omega \subseteq \omega^\omega$, $\dot{s}$ is a name for an element of $\omega^\omega$. Since Miller forcing has continuous reading of names we have a forcing condition $B \subseteq \omega^\omega$ and a continuous function $f : B \to \omega^\omega$ such that

$$D \Vdash \dot{s} = f(\dot{m}),$$

where $\dot{m}$ is the name for the $\mathbb{M}$-generic real. By another well known property of Miler forcing there exists a stronger condition $D \subseteq B$ such that either $f \restriction D$ is constant or $f \restriction D$ is a topological embedding. We can exclude the first possibility. Let us denote $E = f[D]$ and note that since $P \restriction E$ is Borel, there is a dense $G_\delta$ set $G \subseteq E$ such that $P \restriction G$ is continuous. But then $f^{-1}[G]$ is comeager in $D$ and hence is a condition in Miller forcing. But

$$f^{-1}[G] \Vdash \dot{s} \in G$$

and $G \in I_P$ which gives a contradition. $\square$

## 8. A forcing without continuous reading of names in any presentation

The Steprāns forcing notion does not have continuous reading of names in one presentation and has it in another. Let us now show how to use Steprāns forcing to produce a forcing $\mathbb{P}_I$ which is proper and does not have continuous reading of names in any presentation.

**Theorem 9.** *There exist an ideal $I \subseteq Bor(\omega^\omega)$ such that the forcing $\mathbb{P}_I$ is proper but it does not have continuous reading of names in any presentation.*

*Proof.* First notice that any presentation of a Polish space $X$ is given by a Borel isomorphism with another Polish space $Y$ and the latter can be assumed to be a $G_\delta$ subset of $[0,1]^\omega$. Instead of $\omega^\omega$ let us consider $X = (\omega^\omega)^2$ with its product topology. Note that each $G_\delta$ set in $[0,1]^\omega$ as well as a Borel isomorpshism from $\omega^\omega$ to $X$ can be coded by a real. Let $x \in \omega^\omega$ code a pair $(G_x, f_x)$ that defines a presentation of $X$ as above, i.e. $f_x : G_x \to X$. For $x \in \omega^\omega$ $f_x^{-1}[X_x]$ ($X_x$ denotes the vertical section of $X$ at $x$) is an uncountable Borel set in $G_x$ and contains a copy $C_x$ of $(\omega+1)^\omega$. Let $I_x$ be the transported ideal $I_P$ from $C_x$ to $X_x$. We define an ideal $I$ on $\mathrm{Bor}(X)$ as follows:

$$I = \{A \in \mathrm{Bor}(X) : \forall x \in \omega^\omega\ A_x \in I_x\}.$$

$\mathbb{P}_I$ does not have continuous reading of names in any presentation for if $(G_x, f_x)$ defines a presentation then $(P \circ f_x^{-1}) \restriction (f_x[C_x])$ is a counterexample to continuous reading of names in its topology. Let us show that $\mathbb{P}_I$ is proper. Take $H_\kappa$ big enough so that the function $\omega^\omega \ni x \mapsto (G_x, f_x, C_x, I_x)$ is in $H_\kappa$. Take any model $M \prec H_\kappa$ which contains this function and let $B \in M$ be $I$-positive. By elementarity there is $x \in \omega^\omega \cap M$ such that $B_x \notin I_x$. So $B_x \cap f_x^{-1}[C_x]$ is $I_x$-positive. Now since the forcing $\mathbb{P}_I$ below $f_x^{-1}[C_x]$ is equivalent to the Steprāns forcing, which is proper, it follows that there is $B' \subseteq B_x \cap f_x^{-1}[C_x]$ $I_x$-positive and $M$-master.

□

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